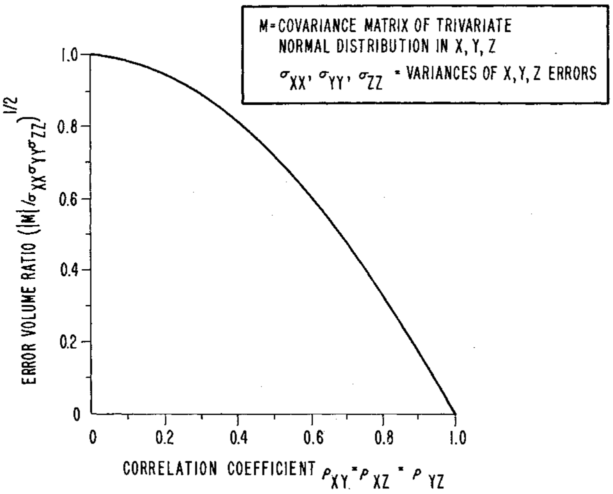

Fig. 1 Effect of correlation on error volume approximation.

In order for the density function (1) to be defined, $|M|>0$; hence it remains only to show that the expression involving the ρ's in parentheses in (3) is nonnegative. This follows from the inequalities

$$\rho_{xy}^2 + \rho_{xz}^2 + \rho_{yz}^2 \geq \rho_{xy}^2 + \rho_{xz}^2 \geq |2\rho_{xy}\rho_{xz}| \geq |2\rho_{xy}\rho_{xz}\rho_{yz}| \quad (4)$$

where use is made of the property that any correlation coefficient, by definition, cannot exceed unity in absolute value. Combining the first and last sections of (4),

$$\rho_{xy}^2 + \rho_{xz}^2 + \rho_{yz}^2 - 2\rho_{xy}\rho_{xz}\rho_{yz} \geq 0 \quad (5)$$

and, since $|M| > 0$, it is concluded from (3) that

$$\sigma_{xx}\sigma_{yy}\sigma_{zz} \geq |M| \quad (6)$$

To give an idea of how the degree of correlation between the three error variables affects the error volumes previously described, the ratio $|M|^{1/2}/(\sigma_{xx}\sigma_{yy}\sigma_{zz})^{1/2}$ obtained from (3) is plotted in Fig. 1 as a function of (positive) correlation coefficient. This ratio is equivalently the ratio of error volumes calculated with and without the covariance elements included in M. For simplicity, the three paired correlation coefficients ρ_{xy}, ρ_{xz}, ρ_{yz} are taken here as equal, but the effect of unequal coefficients can be roughly inferred from these results. Figure 1 indicates that a relatively high degree of correlation must exist between the error variables before the error volume is significantly altered from its approximate value obtained by considering the errors to be independent. For example, a common correlation coefficient of 0.5 reduces the actual error volume to about 0.7 of its approximate value.

Use of the Adjoint System in the Solution of Two-Point Boundary Value Problems

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Some papers[1, 2] have appeared recently on the use of the adjoint system in solving a two-point boundary value problem for a system of n first order differential equations. While the method described is workable, it does not seem to have any pronounced advantage over a more obvious method which makes use of the equations of variation directly and thereby avoids some of the rather confusing aspects of the adjoint equations. The special two-point boundary value problem considered is that of solving the equations

$$\dot{x}^i = f^i(x^1, \ldots, x^n, t) \qquad (i = 1, \ldots, n) \quad (1)$$

subject to the initial conditions

$$x^i(t_0) = a^i \qquad (i = 1, \ldots, r) \quad (2)$$

and the final ($t_1 > t_0$) conditions

$$x^i(t_1) = b^i \qquad (i = r + 1, \ldots, n) \quad (3)$$

Let $y^i(t)$ be the solution (a set of functions) of (1) subject to the initial conditions

$$y^i(t_0) = a^i \qquad (i = 1, \ldots, n)$$

where $a^{r+1}, \ldots, a^n$ are estimates of the unspecified initial conditions which will produce the final conditions (3), and suppose that the final values are found actually to be

$$y^i(t_1) = c^i \qquad (i = r + 1, \ldots, n)$$

If

$$x^i(t) = y^i(t) + \xi^i(t) \quad (4)$$

the equations of variation are

$$\dot{\xi}^i = \sum_{j=1}^{n} f_j^i \xi^j \quad (5)$$

where f_j^i denotes the partial derivative of $f^i(x^1, \ldots, x^n, t)$ with respect to x^j after the solution functions $y^i(t)$ have been substituted for the x^i, so that the f_j^i are known functions of t. The initial and final conditions to be imposed on the solution of (5) are found from (4) to be

$$\xi^i(t_0) = 0 \qquad (i = 1, \ldots, r) \quad (6)$$

and

$$\xi^i(t_1) = b^i - c^i = \beta^i \qquad (i = r + 1, \ldots, n) \quad (7)$$

where the β^i are known. The problem of solving (5) subject to (6) and (7) has the same character as that of solving (1) subject to (2) and (3), but the linearity of Eq. (5) resulting from the neglect of higher order terms can be exploited to obtain a solution. Once the solution has been obtained, the values for $i = r + 1, \ldots, n$ of $\xi^i(t_0)$ can be found and the unspecified initial values for $i = r + 1, \ldots, n$ of $x^i(t_0)$ are approximately $a^i + \xi^i(t_0)$. A new solution y^i is then obtained with these estimates, and the entire process is repeated until convergence is obtained.

Instead of solving Eqs. (5–7) directly, the method of adjoints introduces a new set of variables λ_i satisfying the adjoint equations

$$\dot{\lambda}_i = -\sum_{j=1}^{n} f_i^j \lambda_j \quad (8)$$

and it is easily seen that any pair of solutions of (5) and (8) satisfies the relation

$$\sum_{i=1}^{n} \xi^i(t_0)\lambda_i(t_0) = \sum_{i=1}^{n} \xi^i(t_1)\lambda_i(t_1) \quad (9)$$

Now consider $n - r$ solutions $\lambda_i^k(t)$ for $k = r + 1, \ldots, n$ of the adjoint equations subject to the final conditions

$$\lambda_i^k(t_1) = 0 \qquad (i = 1, \ldots, r)$$
$$\lambda_i^k(t_1) = \delta_i^k \qquad (i = r + 1, \ldots, n)$$

which define the values $\lambda_i^k(t_0)$. One can now write $n - r$ different versions of Eq. (9)

$$\sum_{i=1}^{n} \xi^i(t_0)\lambda_i^k(t_0) = \xi^k(t_1) = \beta^k \qquad (k = r + 1, \ldots, n) \quad (10)$$

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If $\xi^i(t)$ in all of Eqs. (10) is the solution of (5) subject to (6) and (7), all values in the $n - r$ Eqs. (10) are known except $\xi^i(t_0)$ for $i = r + 1, \ldots, n$, and hence these values can be found.

The same result can be obtained with very little more labor without the use of the adjoint equations. Let $\xi_k{}^i(t)$ for $k = r + 1, \ldots, n$ be $n - r$ solutions of (5) subject to the initial conditions

$$\xi_k{}^i(t_0) = 0 \qquad (i = 1, \ldots, r)$$
$$\xi_k{}^i(t_0) = \delta_k{}^i \qquad (i = r + 1, \ldots, n)$$

These solutions determine the values $\xi_k{}^i(t_1)$. If $\mu^{r+1}, \ldots, \mu^n$ are any $n - r$ constants

$$\xi^i(t) = \mu^{r+1}\xi_{r+1}{}^i(t) + \ldots + \mu^n\xi_n{}^i(t) \qquad (11)$$

is a solution of (5) satisfying (6). If this solution is also to satisfy (7)

$$\beta^i = \mu^{r+1}\xi_{r+1}{}^i(t_1) + \ldots + \mu^n\xi_n{}^i(t_1) \qquad (i = r + 1, \ldots, n)$$

These $n - r$ equations can be solved for $\mu^{r+1}, \ldots, \mu^n$, the solution values substituted in (11), and then putting $t = t_0$ for $i = r + 1, \ldots, n$ yields the required values of $\xi^i(t_0)$. Thus, the steps in this method are exactly parallel to those in the adjoint method except that here one additional computation (equivalent to a matrix $\times$ vector multiplication) is necessary after the simultaneous linear equations have been solved. It hardly seems worth the effort of introducing the adjoint system to avoid this simple step, especially in an exposition of principles.

The first use of the adjoint system in problems having a superficial resemblance to that considered here was by Bliss[3, 4] in his work in ballistics during WW I, but in these applications it serves a much more useful purpose. A simple example of this kind is that in which the $\xi^i(t)$ are variations from a normal trajectory due to abnormal initial conditions and an expression for the final value of just one of the $\xi^i(t)$ is required, in terms of arbitrary initial variations of all of the $\xi^i(t)$. The coefficients in this expression can be obtained with only one integration of the adjoint equations, whereas if the same expression were to be obtained by integrating the equations of variation, n integrations would be required.

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Optimum Planar Circular Orbits Transfer

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Introduction

A RECENT note[1] points out that the solution obtained by Jurovics and McIntyre[2] for the minimum time transfer of a constant thrust acceleration vehicle between two coplanar earth circular orbits is in error. It is apparent from the lengthy footnote in Jurovics and McIntyre's paper that they had considerable difficulties in meeting boundary conditions. Their method of solving the variational boundary value problem has been developed independently by this author[3–5] and successfully applied to the problem at hand as well as many other problems. Convergence has been excellent in all problems solved. These results demonstrate that Jurovics and McIntyre's solution is erroneous (the optimal control is continuous).

It is the purpose of this note to exhibit a continuous optimal control for the problem under discussion, which everywhere satisfies the Legendre-Clebsch necessary condition, and to show that a discontinuous control is nonoptimal. The results obtained by Jurovics and McIntyre are not inherent to the method of solution used, which is demonstrably a very good one.

Discussion

The problem was formulated in terms of the (v,γ,h) state, where v is the total velocity, γ the local flight path angle, and h the altitude. Control is embodied in α, the angle between the velocity and thrust vectors. A minimum time transfer from a 300- to a 1000-statute mile circular earth orbit was obtained for a vehicle with a constant ratio of (thrust acceleration)/(initial gravitational acceleration) equal to 1.7343. Pertinent constants are GM (universal gravitational constant $\times$ earth mass) $= 1.408142 \times 10^{16}$ ft^3/sec^2, and R (earth radius) $= 2.09029 \times 10^7$ ft. This transfer corresponds to a ratio of (radial transfer distance)/(initial orbit radius) equal to 0.164377—very nearly the problem considered by Jurovics and McIntyre[2] and Greenley.[1]

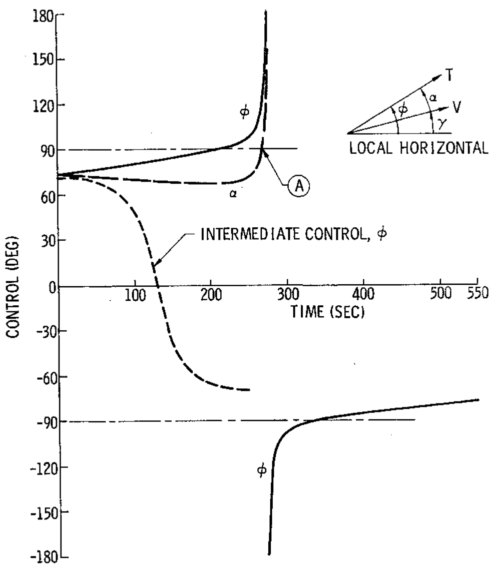

Fig. 1 Optimal control.

The minimum time control obtained is plotted in Fig. 1 as ϕ, the angle between the local horizontal and the thrust direction, which is the control considered by Jurovics and McIntyre. A ratio of (minimum transfer time)/(time per rad in initial orbit) equal to 0.612160 is obtained. The boundary conditions imposed on the variational boundary value problem were met to seven significant figures.

The control α for this problem may be written in terms of the Lagrange multipliers

$$\alpha = \tan^{-1}(\mu_\gamma/v\mu_v) \qquad (1)$$

where μ_v is the multiplier associated with the velocity differential constraint, and μ_γ is the multiplier associated with the path angle differential constraint. It is observed that α is multivalued. The appropriate α may be chosen with the

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